VD cutting VC at C. Produce CD to P making $DC = DP$. Draw the lines AP, BP, and VP. From this construction it easily follows that $\angle BVC = \angle PVA$ and $\angle CVP = 2 \angle CVD$. But $\angle CVP < \angle AVC + \angle PVA$. Hence $\angle CVD < \frac{1}{2}(\angle AVC + \angle BVC)$.

(2) $\angle CVD > 90°$. (Fig. 2.) Produce CV through V and draw plane ABC' as in (1). Then $2 \angle C'VD < \angle AVC' + \angle BVC'$ by (1). Hence $2(180° - \angle C'VD) > 180° - \angle AVC' + 180° - \angle BVC'$ or $\angle CVD > \frac{1}{2}(\angle AVC + \angle BVC)$.

(3) $\angle CVD = 90°$. (Fig. 3.) In this case the plane through AB is parallel to VC. Draw a plane MNV through VD perpendicular to VC and cut by planes BVC and AVC in the lines MV and NV respectively. It can be easily shown that $\angle MVB = \angle NVA$. Hence

$$\angle AVC + \angle BVC = 180°$$

and

$$\angle CVD = \tfrac{1}{2}(\angle AVC + \angle BVC).$$

Also solved by B. LIBBY, F. M. MORGAN, and GEO. W. HARTWELL.

CALCULUS.

338. Proposed by RICHARD LOCHNER, Philadelphia, Pa.

An elliptical field has a major axis of 100 feet and a minor axis of 10 feet. A cow is tethered at the end of the major axis and another at the end of the minor axis. If each cow can graze over half the field, how long is the rope of each? What is the area of the portion over which the cows can graze in common?

SOLUTION BY B. F. FINKEL, Drury College.

The central equation of the elliptic field is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let r_1 be the length of rope by which one cow is tethered at the point A_1, the right-hand extremity of the major axis. The equation of the circle over which this cow can browse is $(x - a)^2 + y^2 = r^2$. The coördinates of the point of intersection, P_1, of this circle with the ellipse are

$$(x_1, y_1) \equiv \left(a\frac{[a^2 - \sqrt{b^4 + r^2(a^2 - b^2)}]}{(a^2 - b^2)}, \right.$$

$$\left. \sqrt{2a^4 \frac{\sqrt{b^4 + r^2(a^2 - b^2)} - b^2(a^2 - b^2)r^2 - a^2(a^4 + b^4)}{(a^2 - b^2)}} \right),$$

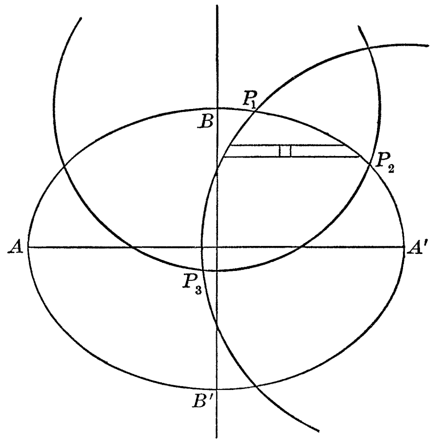
The area common to the ellipse and the circle is

$$\text{area} = 2\int_0^{y_1}\int_{a-\sqrt{r^2-y^2}}^{(a/b)\sqrt{b^2-y^2}} dxdy$$

$$= 2\left[\frac{a}{b}\frac{y_1}{2}\sqrt{b^2 - y_1{}^2} + \frac{b^2}{2}\sin^{-1}\frac{y'}{b} - ay' + \frac{y'}{2}\sqrt{r^2 - y_1{}^2} + \frac{r^2}{2}\sin^{-1}\frac{y_1}{r}\right],$$

which, by the conditions of the problem, $= \frac{1}{2}\pi ab$.

Substituting the value of y_1 in this equation, we have an equation in r. At the cost of great labor, this equation can be solved, by trial, to any degree of accuracy.



In a similar manner, the area over which a cow tethered at B can graze, may be found. Let the coördinates of the points of intersection, P_2, be (x_2, y_2). The length of the radius, BP_2, may be found to any desired degree of accuracy, as above. When this is found x_2 and y_2 become known.

Then to find the area common to the two circles and the ellipse we have only to perform the following integration,

$$\int_{y_3}^{y_2}\int_{a-\sqrt{r_1^2-y^2}}^{\sqrt{r_2^2-(y-b)^2}} dxdy + \int_{y_2}^{y_1}\int_{a-\sqrt{r_1^2-y^2}}^{(a/b)\sqrt{b^2-y^2}} dxdy,$$

where y_3 is the ordinate of P_3.

The case when the circle whose center is B cuts the ellipse in four points, would require still more labor.

343. Proposed by C. N. SCHMALL, New York City.

Show that the envelope of the system of circles whose radii are the ordinates of an ellipse is a concentric ellipse having the same minor axis as the given ellipse.

SOLUTION BY I. A. BARRETT and F. C. REISLER, University of Chicago.

Choose the axes so that the equation of the ellipse is in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Since the radii of the circles are to be ordinates of the ellipse, we have

$$(X-x)^2+Y^2=\frac{b^2}{a^2}(a^2-x^2), \tag{1}$$

where (X, Y) are running coördinates of the circle.